On the reflection point where light reflects to a known destination on quadratic surfaces

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We address the problem of determining the reflection point on a specular surface where a light ray that travels from a source to a target is reflected. The specular surfaces considered are those expressed by a quadratic equation. So far, there is no closed form explicit equation for the general solution of this determination of the reflection point, and the usual approach is to use the Snell law or the Fermat principle whose equations are derived in multidimensional nonlinear minimizations. We prove in this Letter that one can impose a set of three restrictions to the reflection point that can impose a set of three restrictions that culminates in a very elegant formalism of searching the reflection point in a unidimensional curve in space. This curve is the intersection of two quadratic equations. Some applications of this framework are also discussed. © 2010 Optical Society of America

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The reflection of the light in a specular surface is a well-known and studied phenomenon expressed by the Snell law that states that at the reflection point on a perfect specular surface the incident and the reflected rays make an angle of equal amplitude with the normal vector. It also states that normal, incident, and reflected vectors are coplanar [1,2].

Additionally, the Fermat principle states that light travels through the quickest path, which is the same as saying that the light travels through the shortest path in a non-relativistic three-dimensional (3D)-space. We consider no effects of light scattering.

The problem that we solve in this Letter is the determination of the point on the mirror surface that reflects the incident light from a known source to a particular target. This can be addressed as the search of the reflection point for which we know the light emitter and detector positions in relation to the specular surface. There is no closed-form expression for this point, and the best we could do so far is to solve the Snell law or the Fermat principle equations derived in a multidimensional minimization set of equations. However, there is no proof, as far as the author is aware of, that the solution cannot be expressed by an explicit expression. It is thus implicit, nonlinear, often unstable, and computationally demanding.

In computer graphics and vision applications there are two standard solutions using the Snell law. The usual way to do this is to use a nonlinear minimization method to optimize the law equation. Another approach is to use topological nets (called mesh grids) to approximate the mirror surface. The solution for the reflection point is computed as a bilinear interpolation of points at the vicinity of the optimal solution (in smoothly curved mirrors) [3–6]. The accuracy of these approaches depends on the density of photon emitting and on the density of the mirror surface sampling process. They are both based on two-dimensional spaces and on the Snell law (the classical property of reflection).

Furthermore, another approach is based on the Fermat principle, which states that the light takes the shortest path between two points. As noticed by [7] it is an optimization problem in a two-dimensional space. The solution is obtained by taking the derivative of the surface expression to find the minimum of the distance from the source to the target.

We then prove that the determination of the reflection point can be much improved by searching in a unidimensional space rather than a multidimensional one. Particularly, we prove in this Letter that the solution belongs to an elegant quartic curve derived in the intersection of two quadratic equations. The determination of the very point that reflects the light from its source to the target is then searched for in that curve. This algorithm is easy, much more stable, and much quicker. These improvements on the determination of the reflected point on the mirror surface can be of a great help in problems involving optical calibration, calibration of vision systems (machine vision applications), and rendering of complex accurate images (computer graphics).

Homogeneous coordinates are used and points are expressed as $\mathbf{X}=[x_1\ x_2\ x_3\ x_4]^T$. The corresponding Cartesian coordinates are given by $\mathbf{x}=[x\ y\ z]^T$, where $x=x_1/x_4$, $y=x_2/x_4$, and $z=x_3/x_4$. Quadric surfaces are expressed by a 4×4 symmetric matrix $\mathbf{Q}$. A point $\mathbf{X}$ belongs to a quadric surface $\mathbf{Q}$ if and only if it respects the equation $\mathbf{X}^T\mathbf{Q}\mathbf{X}=0$.

It is also easy to prove that the plane coordinates defined by three noncollinear points can be expressed as a linear equation on the coordinates of one of them. Consider the points $\mathbf{U}$, $\mathbf{V}$, and $\mathbf{W}$ that define the plane $\mathbf{\Pi}$. Consider now a fourth point, also belonging to the plane $\mathbf{\Pi}$, expressed by its coordinates $[x\ y\ z\ 1]^T$. Since this fourth point is a linear combination of the other three, the determinant of a matrix stacking the four points must be null. This equation can be regarded as the plane equation, and it is possible to express the plane coordinates as $\mathbf{\Pi}=\mathbf{M}(\mathbf{U},\mathbf{V})\mathbf{W}$, where the matrix $\mathbf{M}(\mathbf{U},\mathbf{V})$ is given by

$$\mathbf{M}(\mathbf{U},\mathbf{V}) = \begin{bmatrix} 0 & u_3v_4 - u_4v_3 & -u_2v_4 + u_4v_2 & u_2v_3 - u_3v_2 \\ -u_3v_4 + u_4v_3 & 0 & u_1v_4 - u_4v_1 & -u_1v_3 + u_3v_1 \\ u_2v_4 - u_4v_2 & -u_1v_4 + u_4v_1 & 0 & u_1v_2 - u_2v_1 \\ -u_2v_3 + u_3v_2 & u_1v_3 - u_3v_1 & -u_1v_2 + u_2v_1 & 0 \end{bmatrix}. \quad (1)$$

Let us consider a light source **S** and a light detector (target) **T**. A light ray emitted by the source **S** is then reflected to the target **T** and its reflection point over the mirror surface is the point **R** (see Fig. 1). The absolute dual quadric ($\mathbf{Q}_\infty$) in the projective geometry is defined as the quadric whose equation is $x^2+y^2+z^2=0$ [8].

The problem we tackle in this Letter is how to find the reflection point **R** that reflects a light ray emitted by a source **S** to a target point **T**. The first step to solve the problem is to characterize the reflection point.

For the point **R** the following three restrictions must be imposed:

Restriction 1: $\mathbf{R}^T\mathbf{Q}\mathbf{R}=0 \rightarrow$ the point is on the quadric of the mirror surface.

Restriction 2: $\mathbf{R}^T\mathbf{A}\mathbf{R}=0 \rightarrow$ the point is on an analytical quadric given by $\mathbf{A}=\mathbf{M}^T\mathbf{Q}_\infty^*\mathbf{Q}$ (proposition 1). Geometrically this constraint means that the normal to the quadric **Q** at point **R** is contained in the plane defined by **S**, **T**, and **R**.

Proposition 1. *The reflection point* **R** *on a quadric mirror* **Q***, reflecting a light ray emitted by a source* **S** *to a target* **T***, is on the quadric surface* **A** *given by* $\mathbf{A}=\mathbf{M}^T\mathbf{Q}_\infty^*\mathbf{Q}$*, where* $\mathbf{Q}_\infty^*$ *is the absolute dual quadric; the* 4×4 *matrix* **M** *is given by expression (1) for the source and target points:* $\mathbf{M}=\mathbf{M}(\mathbf{S},\mathbf{T})$.

Proof. The normal plane to the quadric at the reflection point **R** is perpendicular to the plane defined by the source, the target, and the reflection point itself. These planes are defined by their plane coordinate vectors $\mathbf{\Pi_1}=\mathbf{QR}$ and $\mathbf{\Pi_2}=\mathbf{M}(\mathbf{S},\mathbf{T})\mathbf{R}=\mathbf{MR}$, respectively.

The coordinate vectors of normals to these planes in Euclidean space are $\pi_1=\mathbf{Q}_\infty^*\mathbf{QR}$ and $\pi_2=\mathbf{Q}_\infty^*\mathbf{MR}$. Since they are perpendicular, their scalar product is zero, expressed by $(\mathbf{Q}_\infty^*\mathbf{MR})^T(\mathbf{Q}_\infty^*\mathbf{QR})=0 \Leftrightarrow \mathbf{R}^T\mathbf{M}^T\mathbf{Q}_\infty^{*T}\mathbf{Q}_\infty^*\mathbf{QR}=0$. Since $\mathbf{Q}_\infty^{*T}=\mathbf{Q}_\infty^*$ and $\mathbf{Q}_\infty^{*2}=\mathbf{Q}_\infty^*$, the expression yields

$$\mathbf{R}^T\mathbf{M}^T\mathbf{Q}_\infty^*\mathbf{QR}=0, \quad (2)$$

which is the same as saying that **R** belongs to the quadric $\mathbf{A}=\mathbf{M}^T\mathbf{Q}_\infty^*\mathbf{Q}$. ■

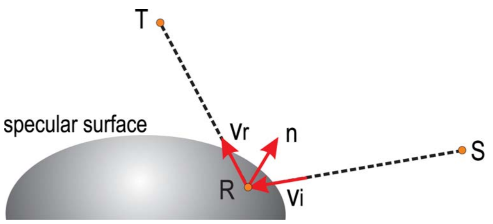

Fig. 1. (Color online) Light ray reflection by a quadratic surface.

Restriction 3: The incident and reflected angles are equal or the sum of distances $\|\mathbf{SR}\|+\|\mathbf{RT}\|$ is a minimum.

This third restriction imposes the choice of **R** on the subspace derived by the previous two constraints. For this particular choice of the reflection point one can use a reasoning based on one of the physical laws.

If the Snell law reasoning is used, for a given point on the mirror surface, the normal vector should make an equal angle with both the incident and reflected rays. This computation is straightforward, and the normal vector can be computed using the coordinates of the tangent plane to the quadric surface at the reflection point **R** such that $\mathbf{\Pi_N}=\mathbf{QR}$. As the normal vector is only the direction of the normal plane, its coordinates are the first three plane coordinates. We have $\mathbf{n}=[\mathbf{I_3}|\mathbf{0}]\Pi_\mathbf{N}$, where $\mathbf{I_3}$ is the 3×3 identity matrix. The reflection point where incident and reflected rays are equal is then the solution of the following expression:

$$\cos\left(\frac{(\mathbf{s}-\mathbf{r})^T}{\|\mathbf{s}-\mathbf{r}\|}\cdot\mathbf{n}\right)=\cos\left(\frac{(\mathbf{t}-\mathbf{r})^T}{\|\mathbf{t}-\mathbf{r}\|}\cdot\mathbf{n}\right), \quad (3)$$

where **s**, **t**, and **r** are the Cartesian coordinates of the corresponding points in capital letters.

The alternative formulation of this third restriction is by using the Fermat principle reasoning, that is, the total distance traveled by the light from the point **S** to the point **T** passing by the reflection point **R** must be minimized. This restriction is expressed by $\min_\mathbf{R}\{\|\mathbf{SR}\|+\|\mathbf{RT}\|\}$.

Both formulations of the third restriction can be used. We observed in experiments that the reasoning based on the Fermat principle has a better performance while maintaining the accuracy.

The three restrictions above can then be used to compute the reflection point on the mirror surface. We notice that we are considering only geometrical solutions in the quadric surface.

Given the three constraints imposed to the reflection point **R**, the problem now is how to determine that point. Its explicit closed form computation is, however, still not possible. The first and second constraints restrict the point **R** to be on quadrics **Q** and **A**. This is the problem of finding the intersection of two quadrics. The third restriction constrains the point so that the incident and reflection angles are equal (Snell law reasoning) or alternatively so that the total distance traveled by the light is a minimum (Fermat principle reasoning). The point **R** must thus be located on the intersection curve where the third restriction is met.

The general method for computing an explicit parametric curve of the intersection between two quadrics is due to Levin [9]. However, the parametric representation of this method is hard to compute and is less reliable owing to the high number of irrationals. Dupont *et al.* [10–12] presented a modification of the Levin method to intersect quadrics, demonstrating that this alternative method is much more accurate than the original one.

Although the unicity of the solution is not proved, since nonruled quadric mirrors have a constant curvature, one should expect a single solution. Experimentally a single solution was always observed in the search for **R** on the intersection curve. This topic shall be considered in future developments.

This method of determining the reflection point **R** on a mirror surface that reflects a light ray emitted by a source **S** to a target **T** presents a major advantage over the method of using explicitly the Euclidean expressions of the mirror either using the Snell law or the Fermat principle. This advantage is the fact that, once intersected the quadrics **Q** and **A**, the solution is given by a nonlinear equation in only one parameter. This is important for the accuracy of the solution and also to the computational efficiency of the method, since the intersection of two quadrics can be determined by a noniterative method (see [10] for details).

To validate our method, we performed some experiments comparing the performance of all three methods (for a certain level of accuracy). We thus measured the algorithm time to compute the reflection point for the three considered methods for 400 points. The experiments were performed using MATLAB.

Figure 2 plots the median time for a hyperbolic mirror (diameter of 50 mm) for a specified imposed accuracy (the maximum error in millimeters is the horizontal axis). It can be observed that, as expected, the determination of the reflection point using our method has a performance six to ten times better than the Snell law and the Fermat principle, since a unidimensional space is searched for rather than a multidimensional one.

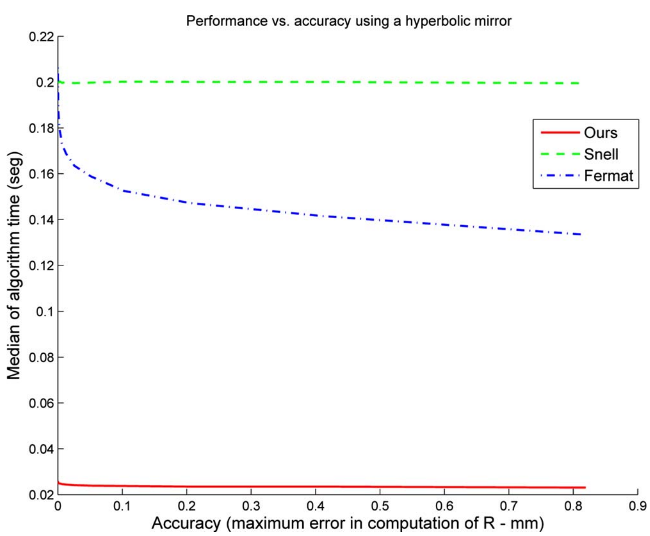

Fig. 2. (Color online) Performance versus accuracy using a hyperbolic mirror and random source and target points.

Concerning the applications, there are several problems that can benefit from the easiest way to determine the reflection point. Panoramic vision systems are composed of curved mirrors (often expressed by quadratic surfaces) and perspective cameras. Their calibration is not trivial and can highly benefit from the existence of expressions that relates 3D points with the corresponding image points. Using our derivations to determine the reflection point, the calibration and even the 3D reconstruction can become more efficient and accurate.

Another strong application of this framework is the industry of computer graphics. Particularly, the rendering of virtual images for advertising, movies, or games is computationally heavy owing to billions of calculations. Environments with mirrors and other specular materials are very popular and produce attractive images. The framework presented in this Letter is able to greatly increase the performance of the rendering algorithm.

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